

RSOS models and Jantzen-Seitz representations of Hecke algebras at roots of unity.

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Abstract

A special family of partitions occurs in two apparently unrelated contexts: the evaluation of 1-dimensional configuration sums of certain RSOS models, and the modular representation theory of symmetric groups or their Hecke algebras H_m . We provide an explanation of this coincidence by showing how the irreducible H_m -modules which remain irreducible under restriction to H_{m-1} (Jantzen-Seitz modules) can be determined from the decomposition of a tensor product of representations of $\widehat{\mathfrak{sl}}_n$.

1 Introduction

The solution of a class of “restricted-solid-on-solid” (RSOS) models by the corner transfer matrix method leads to the evaluation of weighted sums of combinatorial objects called paths [1]. The Kyoto group realized that these combinatorial sums are branching functions of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ [6], and was able to define similar models associated with other affine Lie algebras, in particular to $\widehat{\mathfrak{sl}}_n$ [16].

For the models associated with the cosets $(\widehat{\mathfrak{sl}}_n)_1 \times (\widehat{\mathfrak{sl}}_n)_1 / (\widehat{\mathfrak{sl}}_n)_2$, a different description of the branching functions as generating series of certain sets of partitions has been obtained in [9], and was used to derive fermionic expressions for the configuration sums.

It turns out that exactly the same partitions arise in the modular representation theory of the symmetric groups: as conjectured by Jantzen and Seitz [14] and established recently by Kleshchev [20], such partitions label the irreducible representations of a symmetric group \mathfrak{S}_m over a field of characteristic n which remain irreducible under restriction to \mathfrak{S}_{m-1} .

The aim of this Letter is to provide an explanation of this seemingly mysterious coincidence.

The first point is to replace symmetric groups in characteristic n by Hecke algebras of type A over \mathbb{C} at an n th root of unity. Indeed, the representation theories of $\mathbb{F}_n[\mathfrak{S}_m]$ and $H_m(\sqrt[n]{1})$ have many formal similarities, but the consideration of Hecke algebras removes the restriction of n being a prime, which does not appear on the statistical mechanics side.

Moreover, a connection between the representation theory of $H_m(\sqrt[n]{1})$ and the level 1 representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$ has been pointed out in [22]. Building on a conjecture of [22], recently proved by Ariki and Grojnowski, we show that the Jantzen-Seitz type problem for $H_m(\sqrt[n]{1})$ is equivalent to the decomposition via crystal bases of tensor products of level 1 $\widehat{\mathfrak{sl}}_n$ -modules. Using the results of [9], we can then characterize the Hecke algebra modules of Jantzen-Seitz type and explain the occurrence of their partition labels in the configuration sums of RSOS-models.

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As an application, we express the generating function of the Jantzen-Seitz partitions having a given n -core in terms of branching functions of $\widehat{\mathfrak{sl}}_n$. This is to be compared with a well-known result on blocks of Hecke algebras. Indeed, the blocks of $H_m(\sqrt[n]{1})$ are labelled by n -cores, and the dimension of a block is the number of n -regular partitions of m with the corresponding n -core. Using a formula first proved in the fifties by Robinson (in the symmetric group case) one can compute the generating series of the dimensions of all blocks labelled by a given n -core, and recognize the string function of the level 1 $\widehat{\mathfrak{sl}}_n$ -modules. Our result shows that some branching functions of $\widehat{\mathfrak{sl}}_n$ other than the level 1 string function arise in a natural way in the representation theory of $H_m(\sqrt[n]{1})$.

2 Characters and branching functions of $\widehat{\mathfrak{sl}}_n$

Using the notion of paths, it was shown in [5] that the characters of the integrable highest weight modules of $\widehat{\mathfrak{sl}}_n$ may be obtained by enumerating certain *coloured multipartitions*. In this Letter, we are interested only in the case when the highest weight is of level one, and therefore the multipartitions are simply partitions.

As is usual, we define a partition λ of m to be a sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$$

such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \dots + \lambda_k = m$. If $i > k$, we understand that $\lambda_i = 0$. The set of all partitions of m is denoted $\Pi(m)$ and we write

$$\Pi = \bigcup_{m \geq 0} \Pi(m).$$

Occasionally, it will be convenient to use the exponent notation

$$\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \dots, \lambda_r^{a_r})$$

where here $\lambda_1 > \lambda_2 > \dots > \lambda_r > 0$, and $a_i > 0$ specifies the multiplicity of λ_i in λ . If $a_i < n$ for $1 \leq i \leq r$ then we say that the partition λ is n -regular. The set of all n -regular partitions of m is denoted $\Pi_n(m)$, and we define

$$\Pi_n = \bigcup_{m \geq 0} \Pi_n(m).$$

The Young diagram associated with the partition λ is an array of k left-adjusted rows of nodes (or boxes) in which the i th row contains λ_i nodes. For example if $\lambda = (4, 3, 1)$, the corresponding Young diagram is:

$$F^{(4,3,1)} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}.$$

We will not distinguish between a partition and its Young diagram. The partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ conjugate to λ is defined such that λ'_j is the length of the j th column of λ , reading from left to right.

A coloured partition is a Young diagram in which each node γ is filled by its *colour charge* $c(\gamma)$ given by $c(\gamma) = (j - i) \bmod n$, when γ is the node at the intersection of the i th row and the j th column. For example, in the case $n = 3$, the coloured partition $\lambda = (5, 5, 4, 1, 1)$ appears as follows:

$$\begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & 2 & 0 \\ \hline 1 & 2 & 0 & 1 & \\ \hline 0 & & & & \\ \hline 2 & & & & \\ \hline \end{array}.$$

Let m_i be the number of nodes of λ with colour charge i . The energy of λ is defined by $E(\lambda) = m_0$, and its weight by $\text{wt}(\lambda) = \Lambda_0 - \sum_{i=0}^{n-1} m_i \alpha_i$. (We use freely the standard notations for roots, weights, etc. of $\widehat{\mathfrak{sl}}_n$, see *e.g.* [5].) Then, the formal character of the basic representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$ is

$$\text{ch } V(\Lambda_0) = \sum_{\lambda \in \Pi_n} e^{\text{wt}(\lambda)}.$$

A realisation of $V(\Lambda_0)$ that has a basis naturally indexed by the set of n -regular partitions Π_n was given in [5].

In [15], it was also shown how to describe combinatorially the branching functions of tensor products of irreducible $\widehat{\mathfrak{sl}}_n$ -modules. The branching function $b_{\Lambda, \Lambda'}^{\Lambda''}(q)$ is defined such that, for each $i \geq 0$, the multiplicity of the module $V(\Lambda'' - i\delta)$ in the tensor product $V(\Lambda) \otimes V(\Lambda')$ is given by the coefficient of q^i in $b_{\Lambda, \Lambda'}^{\Lambda''}(q)$. Therefore, in terms of characters, we have

$$\text{ch}(V(\Lambda))\text{ch}(V(\Lambda')) = \sum_{\Lambda'' \in P^+} b_{\Lambda, \Lambda'}^{\Lambda''}(e^{-\delta})\text{ch}(V(\Lambda'')).$$

The result of [15] relies on the notion of a path which we describe in the case where $\Lambda' = \Lambda_0$. A path p is a sequence $p = (p_0, p_1, \dots)$, where $p_k \in P$, the weight lattice of $\widehat{\mathfrak{sl}}'_n$. Given a dominant integral weight Λ of level l for some $l \geq 0$, and $\lambda \in \Pi_n$, a path $p = p(\lambda)$ is determined as follows. For $k \geq \lambda_1$, define $p_k = \Lambda + \Lambda_k$. Then for $0 < k \leq \lambda_1$, recursively obtain p_{k-1} from p_k by

$$p_{k-1} = p_k - \epsilon_{k-1} - \lambda'_k,$$

where $\epsilon_i = \Lambda_{i+1} - \Lambda_i$ and as usual the indices are understood modulo n . We write $\lambda \in \mathcal{Y}(\Lambda, \Lambda_0)$ if and only if all the coordinates p_k of the path $p(\lambda)$ are dominant weights.

Theorem 2.1 [15]

$$b_{\Lambda, \Lambda_0}^{\Lambda'}(q) = \sum_{\substack{\lambda \in \mathcal{Y}(\Lambda, \Lambda_0) \\ \text{wt}(\lambda) = \Lambda' - \Lambda \pmod{\delta}}} q^{E(\lambda)}.$$

In the case when Λ is of level one, the partitions appearing in this sum were characterised in [9]. We define the set $FW(n, j) \subset \Pi_n$ as follows. Let $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \dots, \lambda_r^{a_r}) \in \Pi_n$. Then $\lambda \in FOW(n, j)$ if and only if either $r = 1$ or

$$a_i + \lambda_i - \lambda_{i+1} + a_{i+1} = 0 \pmod{n},$$

for $i = 1, 2, \dots, r-1$ and $j = (\lambda_1 - a_1) \pmod{n}$.

Theorem 2.2 [9]

$$\mathcal{Y}(\Lambda_j, \Lambda_0) = FOW(n, j).$$

If we now define $FW(n, j, k)$ to be the subset of $FW(n, j)$ comprising those partitions λ for which $\text{wt}(\lambda) = \Lambda_k + \Lambda_{j-k} - \Lambda_j \pmod{\delta}$, then Theorem 2.1 immediately yields the following:

Corollary 2.3 [9] *Let $0 \leq j, k < n$. Then the branching function $b_{\Lambda_j, \Lambda_0}^{\Lambda_k + \Lambda_{j-k}}(q)$ is given by:*

$$b_{\Lambda_j, \Lambda_0}^{\Lambda_k + \Lambda_{j-k}}(q) = \sum_{\lambda \in FOW(n, j, k)} q^{E(\lambda)}.$$

By means of this expression, the following was obtained:

Theorem 2.4 [9] *Let $0 \leq j < n$ and $\Lambda = \Lambda_s + \Lambda_t$ with $0 \leq s \leq t < n$. In addition, let C be the Cartan matrix of \mathfrak{sl}_n , e_i be the $(n-1)$ -dimensional unit vector $(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)^t$ and we set $e_n = 0$. Then*

$$b_{\Lambda_j \Lambda_0}^\Lambda(q) = \sum_m \frac{q^{m^t C^{-1} m - m^t C^{-1} e_{s-t+n} + st/n}}{\prod_{i=1}^{n-1} (q)_{m_i}},$$

where the sum is taken over all $m \in (\mathbb{Z}_{\geq 0})^{\times(n-1)}$ satisfying $t + \sum_{i=1}^{n-1} i m_i = 0 \pmod{n}$, and $(q)_k = (1-q)(1-q^2) \cdots (1-q^k)$.

3 Modular representations

The set of partitions $FW(n, j, k)$ has a definite meaning in modular representation theory. Indeed, $\bigcup_{j,k} FW(n, j, k)$ labels the modular representations of \mathfrak{S}_m in characteristic n that remain irreducible under reduction to \mathfrak{S}_{m-1} [14, 20].

However, in the case of $FW(n, j, k)$, n can be any positive integer, whereas in the case of \mathfrak{S}_m , it has to be a prime number. This difference can be removed by working in the context of Hecke algebras at an n th root of unity, where the Jantzen-Seitz problem still makes sense.

The Hecke algebra $H_m(v)$ of type A_{m-1} , is the $\mathbb{C}(v)$ -algebra generated by the elements T_1, \dots, T_{m-1} subject to the relations

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}; \\ T_i T_j &= T_j T_i \quad |i - j| > 1; \\ T_i^2 &= (v - 1)T_i + v. \end{aligned}$$

In the case $v = 1$, $H_m(v)$ may be identified with the group algebra of the symmetric group \mathfrak{S}_m , through identifying T_i with the simple transposition $(i, i+1) \in \mathfrak{S}_m$. In fact, for generic values of v , $H_m(v)$ is isomorphic to the group algebra of the symmetric group \mathfrak{S}_m . Hence, it is semisimple, and its irreducible representations are parametrised by $\Pi(m)$. A convenient realisation of the representation labelled by λ is the Specht module S^λ in which the entries of the representation matrices of the generators are elements of $\mathbb{Z}[v]$.

In the non-generic case when v is a primitive n th root of unity, $H_m(v)$ is no longer semisimple in general, and its representations are not necessarily completely reducible. The full set of irreducible representations is indexed by $\Pi_n(m)$ (see [4]). The irreducible module labelled by $\mu \in \Pi_n(m)$ is denoted by D^μ .

Define $JS(n)$ to be the set of all partitions that label the irreducible representations of $H_m(\sqrt[n]{1})$ which remain irreducible under reduction to $H_{m-1}(\sqrt[n]{1})$. One of the aims of this work is to show that the partitions in $JS(n)$ are defined by the same conditions as in the case of symmetric groups, and to explain why

$$JS(n) = \bigcup_{j,k} FW(n, j, k).$$

To tackle this problem, it is convenient to consider, as was done in [22, 2], the Grothendieck group $G_0(H_m(\sqrt[n]{1}))$ of the category of finitely generated $H_m(\sqrt[n]{1})$ -modules. The elements of $G_0(H_m(\sqrt[n]{1}))$ are classes $[M]$ of modules, where $[M_1] = [M_2]$ if and only if the simple composition factors of M_1 and M_2 occur with identical multiplicities (the order of the composition factors in the series is disregarded). The sum is defined by $[M] + [N] = [M \oplus N]$. It is known that this is a free abelian group with basis the set $\{[D^\mu]\}$ of classes of irreducible $H_m(\sqrt[n]{1})$ -modules. Then define $\mathcal{G}(n)$ as the direct sum $\mathcal{G}(n) = \bigoplus_m G_0(H_m(\sqrt[n]{1}))$.

As observed in [22], $\mathcal{G}(n)$ can be identified with the basic representation $V(\Lambda_0)$ of $\widehat{\mathfrak{sl}}_n$, the action of the Chevalley generators e_i, f_i being given by the i -restriction and i -induction operators, as defined in the fifties by G. de B. Robinson in the case of symmetric groups.

As an integrable highest weight module of an affine algebra, $\mathcal{G}(n)$ has a canonical basis in the sense of Lusztig and Kashiwara. It turns out that this canonical basis coincides with the natural basis given by the classes $[D(\mu)]$ of the irreducible $H_m(\sqrt[n]{1})$ -modules.

To prove this, and to compute explicitly the canonical basis, one considers the Fock space representation \mathcal{F} of $\widehat{\mathfrak{sl}}_n$, which contains $V(\Lambda_0)$ as its highest irreducible component:

$$\mathcal{F} = V(\Lambda_0) \oplus V_{\text{low}} .$$

The standard basis (v_λ) of \mathcal{F} is labelled by all $\lambda \in \Pi$ and one has an $\widehat{\mathfrak{sl}}_n$ -homomorphism

$$\begin{aligned} d : \mathcal{F} &\longrightarrow V(\Lambda_0) \simeq \mathcal{F}/V_{\text{low}} \\ v_\lambda &\longmapsto v_\lambda \bmod V_{\text{low}} \end{aligned}$$

If \mathcal{F} is identified with the Grothendieck ring of all $H_m(v)$ for a generic v by writing $[S^\lambda] = v_\lambda$, then d coincides with the decomposition map of modular representation theory. In order to introduce the canonical basis, one needs to q -deform the picture and to consider the q -Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$.

4 The Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$

The q -Fock space \mathcal{F} of $U_q(\widehat{\mathfrak{sl}}_n)$ has been described in [11] using a q -analogue of the Clifford algebra. Another realization was given in [24, 19] in terms of semi-infinite q -wedge products. Let (v_λ) denote the standard basis of weight vectors of \mathcal{F} . The Fock space \mathcal{F} affords an integrable representation of $U_q(\widehat{\mathfrak{sl}}_n)$ whose decomposition into irreducible highest weight modules is given by

$$\mathcal{F} = \bigoplus_{k \geq 0} V(\Lambda_0 - k\delta)^{\oplus p(k)} . \quad (1)$$

In [23], the lower crystal basis of \mathcal{F} and its crystal graph structure was described. Let $A \subset \mathbb{Q}(q)$ denote the ring of rational functions without a pole at $q = 0$. The lower crystal basis at $q = 0$ of \mathcal{F} is the pair (L, B) where L is the lower crystal lattice given by

$$L = \bigoplus_{\lambda \in \Pi} A v_\lambda ,$$

and B is the basis of the \mathbb{Q} -vector space L/qL given by

$$B = \{v_\lambda \bmod qL \mid \lambda \in \Pi\} .$$

The action of Kashiwara's operators \tilde{f}_i on the element $v_\lambda \in B$ corresponds to adding to λ a certain node of colour i which is called a good addable i -node. Likewise the action of \tilde{e}_i corresponds to the removal from λ of a certain node of colour i which is called a good removable i -node. As observed in [22], these nodes are precisely those used by Kleshchev in his modular branching rule for symmetric groups [21], whence the terminology.

For each $\lambda \in \Pi$, the largest integer k such that $\tilde{e}_i^k(\lambda) \neq 0$ (*resp* $\tilde{f}_i^k(\lambda) \neq 0$) is denoted by $\varepsilon_i(\lambda)$ (*resp* $\varphi_i(\lambda)$). The crystal graph Γ_n of \mathcal{F} is disconnected and reflects the decomposition (1). The connected component of the empty partition is the crystal graph of $V(\Lambda_0)$ and its vertices are labelled by Π_n .

We denote by $\{G(\mu), \mu \in \Pi_n\}$ the lower global basis of $V(\Lambda_0)$. It is the $\mathbb{Q}[q, q^{-1}]$ -basis of the integral form $V_{\mathbb{Q}}(\Lambda_0)$ of $V(\Lambda_0)$ characterized by

$$G(\mu) \equiv v_\mu \pmod{qL}, \quad \overline{G(\mu)} = G(\mu).$$

Here, for $v = xv_\emptyset \in V(\Lambda_0)$, \bar{v} is defined by $\bar{v} = \bar{x}v_\emptyset$, where $x \mapsto \bar{x}$ is the $\mathbb{Q}(q)$ -ring automorphism of $U_q(\widehat{\mathfrak{sl}}_n)$ given by

$$\bar{q} = q^{-1}, \quad \bar{q^h} = q^{-h}, \quad \bar{e_i} = e_i, \quad \bar{f_i} = f_i,$$

for all $h \in P^\vee$ and $0 \leq i < n$.

The upper global crystal basis $\{G^{up}(\mu), \mu \in \Pi_n\}$ of $V(\Lambda_0)$ [17] is the basis adjoint to $\{G(\mu)\}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle v_\emptyset, v_\emptyset \rangle = 1 \quad \langle q^h v, v' \rangle = \langle v, q^h v' \rangle, \quad \langle e_i v, v' \rangle = \langle v, f_i v' \rangle$$

for all $v, v' \in V(\Lambda_0)$, $h \in P^\vee$ and $0 \leq i < n$.

From now on we fix $n \geq 2$ and write H_m instead of $H_m(\sqrt[n]{1})$. It has been conjectured in [22] and proved in [2] that at $q = 1$, the upper global crystal basis of the $U_q(\widehat{\mathfrak{sl}}_n)$ -module $V(\Lambda_0)$ coincides in the identification $V(\Lambda_0) \simeq \mathcal{G}(n)$ with the basis $([D^\lambda])$ of $\mathcal{G}(n)$. This was also proved by Grojnowski, using the results of [10]. This implies that

Theorem 4.1 *If $\lambda \in \Pi_n(m)$ and the coefficients $c_{\lambda\mu}(q)$ are defined by:*

$$\left(\sum_{i=0}^{n-1} e_i \right) G^{up}(\lambda) = \sum_{\mu \in \Pi_n(m-1)} c_{\lambda\mu}(q) G^{up}(\mu),$$

then,

$$[D^\lambda \downarrow_{H_{m-1}}^{H_m}] = \bigoplus_{\mu \in \Pi_n(m-1)} c_{\lambda\mu}(1) [D^\mu].$$

5 Jantzen-Seitz modules of Hecke algebras

We are now in position to prove the following result.

Theorem 5.1 *Let $\lambda = (\lambda_1^{a_1}, \dots, \lambda_r^{a_r}) \in \Pi_n(m)$. Then $D^\lambda \downarrow_{H_{m-1}}^{H_m}$ is irreducible, i.e. $\lambda \in JS(n)$, if and only if either $r = 1$ or $a_i + \lambda_i - \lambda_{i+1} + a_{i+1} \equiv 0 \pmod{n}$ for $i = 1, 2, \dots, r-1$.*

Proof: By Theorem 4.1, $D^\lambda \downarrow$ is irreducible if and only if $e_i G^{up}(\lambda) = G^{up}(\nu)$ for some ν . It is known (see [18], eqs. 5.3.8. – 5.3.10.) that

$$e_i G^{up}(\lambda) = [\varepsilon_i(\lambda)]_q G^{up}(\tilde{e}_i \lambda) + \sum_{\mu \neq \lambda} E_{\lambda\mu}^i(q) G^{up}(\mu) \quad (2)$$

where $E_{\lambda\mu}^i(q)$ is a Laurent polynomial invariant under $q \mapsto q^{-1}$ such that

$$E_{\lambda\mu}^i(q) \in q^{2-\varepsilon_i(\lambda)} \mathbb{Z}[q].$$

If $\varepsilon_i(\lambda) = 1$, then $[\varepsilon_i(\lambda)]_q = 1$, and $E_{\lambda\nu}^i(q) = 0$ since for all k the coefficients of q^k and q^{-k} in $E_{\lambda\mu}^i$ are equal. Therefore $e_i G^{up}(\lambda) = G^{up}(\nu)$ where $\nu = \tilde{e}_i \lambda$. On the other hand, if $e_i G^{up}(\lambda) = G^{up}(\nu)$ for some ν , then (2) implies that $\varepsilon_i(\lambda) = 1$ so that $\nu = \tilde{e}_i \lambda$.

Hence we have obtained that $D^\lambda \downarrow$ is irreducible if and only if $\varepsilon_i(\lambda) = 1$ for some $i \in \{0, 1, \dots, n-1\}$ and $\varepsilon_j(\lambda) = 0$ for $j \neq i$.

Using crystal graph theory (see *e.g.* Lemma 5.1 of [15]), we see that the coefficient of z^d in the branching function $b_{\Lambda', \Lambda_0}^\Lambda(z)$ is the number of vertices $b \in B$ for which $\text{wt}(b) = \Lambda - d\delta$ and $\varepsilon_j(b) \leq \langle \Lambda', h_j \rangle$ for $j = 0, 1, \dots, n-1$. Therefore, in the case $\Lambda' = \Lambda_i$, the coefficient of z^d in $b_{\Lambda_i, \Lambda_0}^\Lambda(z)$ is given by the number of vertices b of the crystal graph of $V(\Lambda_0)$ such that

$$\text{wt}(b) = \Lambda - \Lambda_i - d\delta, \quad \varepsilon_i(b) \leq 1, \quad \varepsilon_j(b) = 0 \quad (j \neq i).$$

It follows from this discussion that the partitions λ such that $D^\lambda \downarrow$ is irreducible can be identified with the set of vertices b of B which contribute to the tensor product branching function $b_{\Lambda_i, \Lambda_0}^\Lambda(z)$ for some i and some Λ . Note that this branching function is non-zero only if $\Lambda = \Lambda_k + \Lambda_{i-k}$ for some k . Then, by 2.3, the Theorem is proved.

Example 5.2 The argument considered in the above proof may be illustrated by reference to the \emptyset -connected component of the crystal graph Γ_3 of $\widehat{\mathfrak{sl}}_3$ given (up to weight 8) in Fig. 1. Here, those partitions λ for which $\lambda \in JS(n)$ have been highlighted with an asterisk. As in the above proof, these partitions correspond to the nodes b in this crystal graph for which, for some j , $\varepsilon_j(b) \leq 1$ and $\varepsilon_i(b) = 0$ for all $i \neq j$.

Now define the set $JS(n, \mu, d)$ to be the subset of $JS(n)$ comprising those partitions with n -core μ and n -weight d (see [13, 7] for definitions of n -core and n -weight). Then define the generating series

$$\chi_{n,\mu}(z) = \sum_{d \geq 0} \#JS(n, \mu, d) z^d.$$

Theorem 5.3 (i) *If $\lambda \in JS(n)$ then the n -core μ of λ is a rectangular partition $\mu = (k^l)$ such that $k + l \leq n$ (it is assumed here that if either $k = 0$ or $l = 0$ then (k^l) means the empty partition).*

(ii) *If $k \neq 0 \neq l$ then*

$$\chi_{n,(k^l)}(z) = z^{-s} b_{\Lambda_{k-l} \Lambda_0}^{\Lambda_k + \Lambda_{-l}}(z),$$

where $s = \min(k, l)$.

(iii)

$$\chi_{n,\emptyset}(z) = \left(\sum_{k=0}^{n-1} b_{\Lambda_k \Lambda_0}^{\Lambda_k + \Lambda_0}(z) \right) - (n-1).$$

Proof: In the tensor product $V(\Lambda_i) \otimes V(\Lambda_0)$, all highest weights are of the form $\Lambda_k + \Lambda_{-l} - e\delta$ with $k - l = i \bmod n$ (for later convenience we use $-l$ and not l here). We take $0 \leq k, l < n$ here and can also assume that $k \leq -l \bmod n$ whereupon $k + l \leq n$, and $l = 0$ only if $k = 0$. By definition, the multiplicity of $V(\Lambda_k + \Lambda_{-l} - e\delta)$ is given by the coefficient of z^e in $b_{\Lambda_i, \Lambda_0}^{\Lambda_k + \Lambda_{-l}}(z)$ and hence, by Corollary 2.3, by the number of partitions $\lambda \in JS(n)$ for which $\text{wt}(\lambda) = \Lambda_k + \Lambda_{-l} - \Lambda_i - e\delta$. We claim that the n -core of such a λ is the rectangular partition $\mu = (k^l)$, for which we calculate $\text{wt}(\mu) = \Lambda_k + \Lambda_{-l} - \Lambda_{k-l} - s\delta = \Lambda_k + \Lambda_{-l} - \Lambda_i - s\delta$, where $s = \min(k, l)$ is the multiplicity of the colour charge 0 in μ . This follows from the fact that for every string of weights $\Lambda, \Lambda - \delta, \Lambda - 2\delta, \dots$, of the $\widehat{\mathfrak{sl}}_n$ -module $V(\Lambda_0)$ (where $\Lambda + \delta$ is not a weight of $V(\Lambda_0)$), those partitions having these weights have the same n -core, and this n -core has weight Λ . Then since $\mu = (k^l)$ with $k + l \leq n$ is manifestly an n -core, it follows that it is the n -core of λ , thence proving part (i).

Part (ii) follows immediately since in the case $k \neq 0$ (so that $l \neq 0$), the partitions enumerated by the branching function $b_{\Lambda_{k-l} \Lambda_0}^{\Lambda_k + \Lambda_{-l}}$ are precisely those elements $\lambda \in JS(n)$ having weight $\text{wt}(\lambda) = \Lambda_k + \Lambda_{-l} - \Lambda_{k-l} - e\delta$, for some e , and hence n -core (k^l) .

Finally, for $k = 0$ and arbitrary l , each partition counted by $b_{\Lambda_{-l} \Lambda_0}^{\Lambda_0 + \Lambda_{-l}}$ has empty n -core, and hence contributes to $\chi_{n,\emptyset}$. However, the empty partition occurs for each l , hence an adjustment of $n-1$ is needed after summing over all l . No other partition λ is repeated since, as indicated by Theorem 2.2, the $b_{\Lambda_{-l} \Lambda_0}^{\Lambda_0 + \Lambda_{-l}}$ to which it contributes is uniquely determined by $-l \bmod n = (\lambda_1 - a_1) \bmod n$. (The summation over $-l$ is replaced by one over k to give the final result).

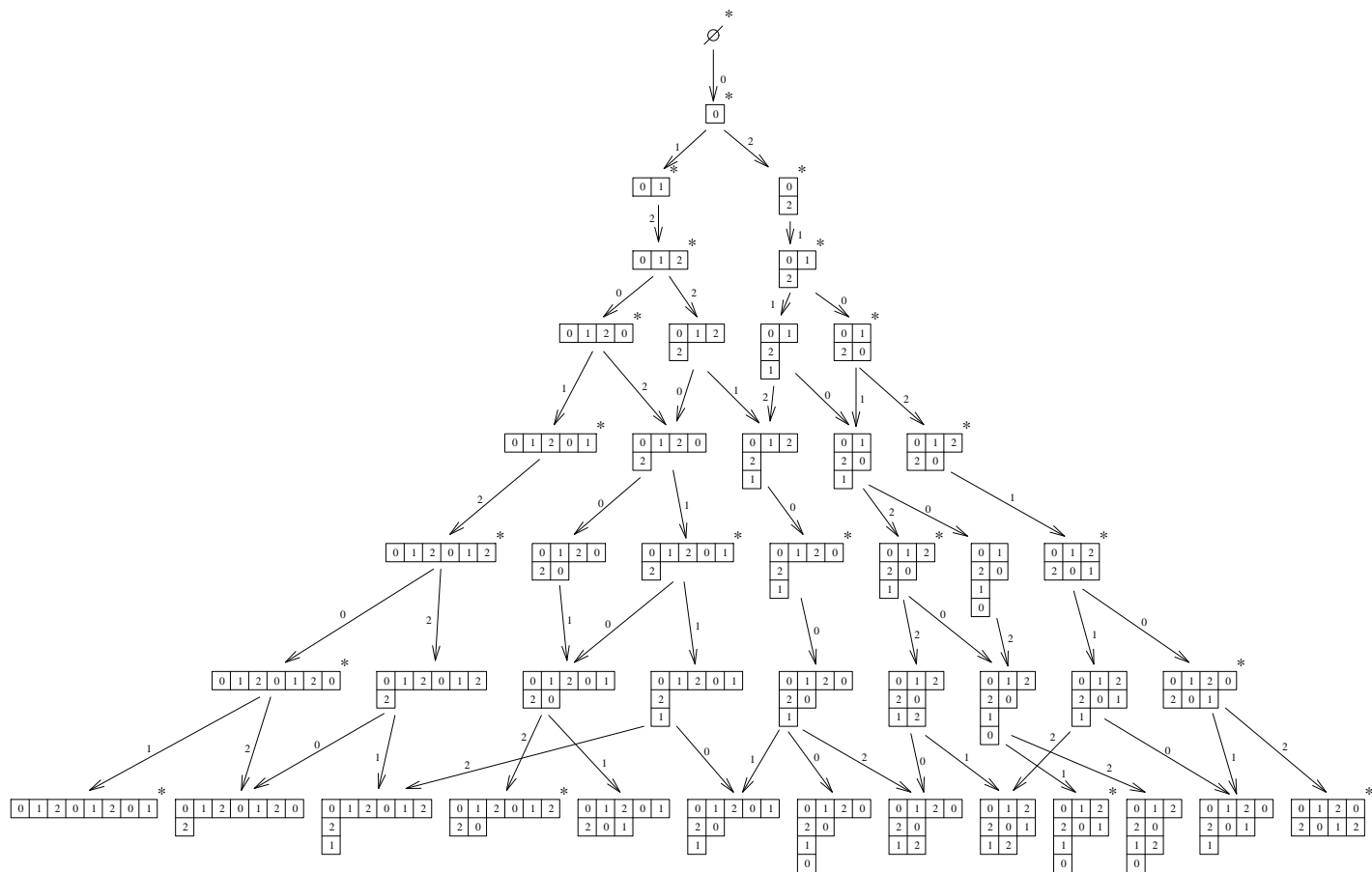


Figure 1.

Example 5.4 To illustrate this result, consider again the case $n = 3$, where we have the following branching functions (to three terms):

$$\begin{aligned}
b_{\Lambda_0, \Lambda_0}^{2\Lambda_0} &= 1 + q^2 + \cdots; \\
b_{\Lambda_0, \Lambda_0}^{\Lambda_1 + \Lambda_2} &= q + 2q^2 + 2q^3 + \cdots; \\
b_{\Lambda_1, \Lambda_0}^{\Lambda_1 + \Lambda_0} &= 1 + q + 2q^2 + \cdots; \\
b_{\Lambda_1, \Lambda_0}^{2\Lambda_2} &= q + q^2 + 2q^3 + \cdots; \\
b_{\Lambda_2, \Lambda_0}^{\Lambda_2 + \Lambda_0} &= 1 + q + 2q^2 + \cdots; \\
b_{\Lambda_2, \Lambda_0}^{2\Lambda_1} &= q + q^2 + 2q^3 + \cdots.
\end{aligned} \tag{3}$$

These are calculated using Corollary 2.3 which leads to the enumeration of the nodes of Fig. 1. labelled by asterisks. The only rectangular 3-cores are \emptyset , (1) , (2) and (1^2) . Using Theorem 5.3, we thus obtain:

$$\begin{aligned}
\chi_{n, \emptyset} &= 1 + 2q + 5q^2 + \cdots; \\
\chi_{n, (1)} &= 1 + 2q + 2q^2 + \cdots; \\
\chi_{n, (2)} &= 1 + q + 2q^2 + \cdots; \\
\chi_{n, (1^2)} &= 1 + q + 2q^2 + \cdots.
\end{aligned} \tag{4}$$

These correspond to the following sets:

$$\begin{aligned}
JS(3, \emptyset, 0) &= \{\emptyset\}; \\
JS(3, \emptyset, 1) &= \{(3), (21)\}; \\
JS(3, \emptyset, 2) &= \{(6), (51), (3^2), (41^2), (321)\}, \\
JS(3, (1), 0) &= \{(1)\}; \\
JS(3, (1), 1) &= \{(4), (2^2)\}; \\
JS(3, (1), 2) &= \{(7), (43)\}, \\
JS(3, (2), 0) &= \{(2)\}; \\
JS(3, (2), 1) &= \{(5)\}; \\
JS(3, (2), 2) &= \{(8), (3^2 1^2)\}, \\
JS(3, (1^2), 0) &= \{(1^2)\}; \\
JS(3, (1^2), 1) &= \{(32)\}; \\
JS(3, (1^2), 2) &= \{(62), (44)\}.
\end{aligned} \tag{5}$$

6 Discussion

We posed and solved the Jantzen-Seitz problem for Hecke algebras of type A . The solution is obtained by mapping the problem to a problem in exactly solvable lattice models, namely that of characterising the space of states of a certain class of restricted solid-on-solid model in terms of the states of the corresponding (unrestricted) solid-on-solid models. The latter was solved in [9]. The relationship between the two problems is based on the fact that both can be formulated in the same language of representation theory of q -affine algebra. The background to these two problems is discussed in greater detail in [7].

In a forthcoming paper [8], we plan to discuss the Jantzen-Seitz problem in the context of type-B Hecke algebras, and more generally, of Ariki-Koike (cyclotomic) Hecke algebras.

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Note added — A forthcoming preprint, [25], contains (among other things) an elementary purely combinatorial proof of part (i) of 5.3.

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